

MODEL THEORY OF A HILBERT SPACE EXPANDED WITH AN UNBOUNDED CLOSED SELFADJOINT OPERATOR

CAMILO ARGOTY

ABSTRACT. We study a closed unbounded self-adjoint operator Q acting on a Hilbert space H in the framework of *Metric Abstract Elementary Classes* (MAECs). We build a suitable MAEC for (H, Γ_Q) , prove it is \aleph_0 -stable up to perturbations and characterize non-splitting and show it has the same properties as non-forking in superstable first order theories. Also, we characterize equality, orthogonality and domination of (Galois) types in that MAEC.

1. INTRODUCTION

This paper deals with a complex Hilbert space expanded by a unbounded closed selfadjoint operator Q , from the point of view of *Metric Abstract Elementary Classes* (see [17]).

Previous works to this paper, can be classified in two kinds. The first one, about model theory of Hilbert spaces expanded with some operators in the frame of continuous logic. The second, about development of a notion of *Abstract Elementary Class* similar to Shelah's (see [23]), but suitable for analytic structures along with its further analysis.

For the first kind, previous work go back to José Iovino PhD Thesis (see [19]), where he and C. W. Henson (his advisor) noticed that the structure $(H, 0, +, \langle \cdot | \cdot \rangle, U)$, where U is a unitary operator, is stable. In [11], Alexander Berenstein and Steven Buechler gave a geometric characterization of forking in that structure after adding to it the projections determined by the Spectral Decomposition Theorem. Ben Yaacov, Usvyatsov and Zadka (see [9]) worked on the first order continuous logic theory of a Hilbert space with a generic automorphism, and characterized the generic automorphisms on a Hilbert space as those whose spectrum is the unit circle. Argoty and Berenstein (see [5]) studied the theory of the structure $(H, +, 0, \langle \cdot | \cdot \rangle, U)$ where U is a unitary operator in the case when the spectrum is countable. The author and Ben Yaacov (see [4]), studied the case of a Hilbert space expanded by a normal operator N . Finally in a recently submitted paper, the author has dealt with non-degenerate representations of an unital (non-commutative) C^* -algebra (see [3]).

For the second kind, in 1980's S. Shelah defined in [23] the so called *Abstract Elementary Class* (AEC) as a generalization of the elementary class which is a class of models of a first order theory. As ever, this paper from Shelah generated a big trend in model theory towards the study of this classes. In order to deal with the case of analytic structures, Tapani Hyttinen and Åsa Hirvonen defined *metric abstract elementary classes* in [17] as a generalization of Shelah's AEC's to classes

The author is very thankful to Alexander Berenstein, Andrés Villaveces and Pedro Zambrano for his help in reading and correcting this work.

of metric structures (MAEC's). After this, in [24, 25] Villaveces and Zambrano studied notions of independence and superstability for *metric abstract elementary classes* (MAEC's).

The main results in this paper are the following:

- We build a MAEC associated with the structure (H, Γ_Q) which is denoted by $\mathcal{K}_{(H, \Gamma_Q)}$.
- We characterize (Galois) types of vectors in some structure in $\mathcal{K}_{(H, \Gamma_Q)}$, in terms of spectral measures.
- We show that $\mathcal{K}_{(H, \Gamma_Q)}$ is \aleph_0 -stable up to perturbations.
- We characterize non-splitting in $\mathcal{K}_{(H, \Gamma_Q)}$ and we show that it has the same properties as non-forking for superstable first order theories.

This paper is divided as follows: In the section 2, we give an introduction to Spectral Theory of unbounded closed selfadjoint operators. In section 3 In this section we define a *metric abstract elementary class* associated with (H, Γ_Q) (denoted by $\mathcal{K}_{(H, \Gamma_Q)}$). In section 4, we give a characterization of definable and algebraic closures. In section 6, we prove superstability of the MAEC $\mathcal{K}_{(H, \Gamma_Q)}$. In section 7, we define spectral independence in $\mathcal{K}_{(H, \Gamma_Q)}$ and we show that it is equivalent to non-splitting with the same properties as non-forking for superstable first order theories. Finally in section 8, we characterize domination, orthogonality of types in terms of absolute continuity and mutual singularity between spectral measures.

2. PRELIMINARIES: SPECTRAL THEORY OF A CLOSED UNBOUNDED SELF-ADJOINT OPERATOR

This is a small review of spectral theory of a closed unbounded self-adjoint operator. The main sources for this section are [15, 21].

Definition 2.1. Let H be a complex Hilbert space. A *linear operator on H* is a function $Q : D(Q) \rightarrow H$ such that $D(Q)$ is a dense vector subspace of H and for all $v, w \in D$ and $\alpha, \beta \in \mathbb{C}$, $Q(\alpha v + \beta w) = \alpha Qv + \beta Qw$.

Definition 2.2. Let Q be a linear operator on H . The operator Q is called *bounded* if the set $\{\|Qu\| : u \in D(Q), \|u\| = 1\}$ is bounded in \mathbb{C} . If Q is not bounded, it is called *unbounded*.

Definition 2.3. If Q is bounded we define the *norm* of S by:

$$\|Q\| = \sup_{u \in D(S), \|u\|=1} \|Su\|$$

For H a Hilbert space, we denote by $B(H)$ the algebra of all bounded linear operators on H .

Definition 2.4. Let R and S be linear operators on H and let $\alpha \in \mathbb{C}$. Then the linear operators $R + S$, αS and S^{-1} are defined as follows:

- (1) If $D(R) \cap D(S)$ is dense in H , $D(R + S) := D(R) \cap D(S)$ and $(R + S)v := Rv + Sv$ for $v \in D(R + S)$.
- (2) $D(RS) := \{v \in H \mid v \in D(S) \text{ and } Sv \in D(R)\}$, $(RS)v := R(Sv)$ if $D(RS)$ is dense and $v \in D(RS)$.
- (3) If $\alpha = 0$, then $\alpha T \equiv 0$ in H . If $\alpha \neq 0$, $D(\alpha S) := D(S)$ and $(\alpha S)v := \alpha Sv$ if $v \in D(S)$.
- (4) If S is one-to-one and $SD(S)$ is dense in H , $D(S^{-1}) := SD(S)$ and $S^{-1}v := w$ if $w \in D(S)$ and $Sw = v$.

Definition 2.5. Let $Q : D(Q) \rightarrow H$ be a linear operator on H . The operator Q is called *closed* if the set $\{(v, Qv) \mid v \in D(Q)\}$ is closed in $H \times H$. The operator Q is called *closable* if the closure of the set $\{(v, Qv) \mid v \in D(Q)\}$ is the graph of some operator which is called the *closure* of Q and is denoted by \bar{Q} .

Definition 2.6. Let Q be an operator (either bounded or unbounded), and λ a complex number

- (1) λ is called a *eigenvalue* of Q if the operator $Q - \lambda I$ is not one to one. The *point spectrum* of Q , denoted by $\sigma_p(Q)$, is the set of all the eigenvalues of Q .
- (2) λ is called a *continuous spectral value* if the operator $Q - \lambda I$ is one to one, the operator $(Q - \lambda I)^{-1}$ is densely defined but is unbounded. The *continuous spectrum* of Q ($\sigma_c(Q)$) is the set of all the continuous spectral values of Q .
- (3) λ is called a *residual spectral value* if $(Q - \lambda I)H$ is not dense in H . The *residual spectrum* of Q ($\sigma_r(Q)$) is the set of all the residual spectral values of Q .
- (4) The *spectrum* of Q ($\sigma(Q)$) is the union of $\sigma_p(Q)$, $\sigma_c(Q)$ and $\sigma_r(Q)$.
- (5) The *resolvent set* of Q ($\rho(Q)$) is the set $\mathbb{C} \setminus \sigma(Q)$. If $\lambda \in \rho(Q)$.
- (6) The *resolvent* of Q at λ is the operator $(Q - \lambda I)^{-1}$, and is denoted by $R_\lambda(Q)$.

Definition 2.7. Given linear operators $Q : D(Q) \rightarrow H$ and $Q' : D(Q') \rightarrow H$ on H , Q' is said to be an *adjoint operator* of Q if for every $v \in D(Q)$ $w \in D(Q')$, $\langle Qv | w \rangle = \langle v | Q^*w \rangle$.

Definition 2.8. Given a linear operator $Q : D(Q) \rightarrow H$ and $Q' : D(Q') \rightarrow H$ on H , then Q' is said to be the *adjoint operator* of Q , denoted Q^* , if Q' is maximal adjoint to Q i.e. if Q'' is an adjoint operator of Q and $Q' \subseteq Q''$ then $Q' = Q''$.

Definition 2.9. An operator Q on H is called *symmetric* if $Q \subseteq Q^*$. If $Q = Q^*$, Q is called *selfadjoint*.

Theorem 2.10 (Lemma XII.2.2 in [15]). *The spectrum of a self adjoint operator Q is real and for $\lambda \in \rho(Q)$, the resolvent $R_\lambda(Q)$ is a normal operator with $R_\lambda(Q)^* = R_{\bar{\lambda}}(Q)$ and $\|R_\lambda(Q)\| \leq |Im(\lambda)|$.*

Theorem 2.11 (Spectral Theorem Multiplication Form, Theorem VIII.4 in [21]). *Let Q be self adjoint on a Hilbert space H with domain $D(Q)$. Then there are a measure space (X, μ) , with μ finite, an unitary operator $U : H \rightarrow L^2(X, \mu)$, and a real function f on X which is finite a.e. so that,*

- (1) $v \in D(Q)$ if and only if $f(\cdot)(Uv)(\cdot) \in L^2(X, \mu)$.
- (2) If $g \in U(D(Q))$, then $(UQU^{-1}g)(x) = f(x)g(x)$ for $x \in X$.

Definition 2.12. A self-adjoint operator Q different from the zero operator is called *positive* and we write $Q \geq 0$, if $\langle Qv | v \rangle \geq 0$ for all $v \in \mathcal{H}$.

Theorem 2.13 (Spectral Theorem-Functional Calculus Form, Theorem VIII.5 in [21]). *Let Q be a closed unbounded self-adjoint operator on H . Then there is a unique map π from the bounded Borel functions on \mathbb{R} into $B(H)$ such that,*

- (1) π is an algebraic $*$ -homomorphism.
- (2) π is norm continuous, that is, $\|\pi(h)\|_{B(H)} \leq \|h\|_\infty$.

- (3) Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of bounded Borel functions with $h_n(x) \rightarrow x$ for each x and $|h_n(x)| \leq |x|$ for all x and n . Then for any $v \in D(Q)$, $\lim_{n \rightarrow \infty} \pi(h_n)v = Qv$.
- (4) Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of bounded Borel functions. If $h_n \rightarrow h$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\pi(h_n) \rightarrow \pi(h)$ strongly.
- (5) If $v \in H$ is such that $Qv = \lambda v$, then $\pi(h)v = h(\lambda)v$.
- (6) If $h \geq 0$, then $\pi(h) \geq 0$

Definition 2.14. Let Ω be a borel measurable subset of \mathbb{R} . By E_Ω we denote the bounded operator $\pi(\chi_\Omega)$ according to Theorem 2.13.

Fact 2.15 (Remark after Theorem VIII.5 in [21]). Previously defined projections satisfy the following properties:

- (1) For every borel measurable $\Omega \subset \mathbb{R}$, $E_\Omega^2 = E_\Omega$ and $E_\Omega^* = E_\Omega$.
- (2) $E_\emptyset = 0$ and $E_{(-\infty, \infty)} = I$
- (3) If $\Omega = \bigcup_{n=1}^\infty \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then $\sum_{n=1}^\infty E_{\Omega_n}$ converges to E_Ω in the strong topology.
- (4) $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$ (and therefore E_{Ω_1} commutes with E_{Ω_2}) for all borel measurable $\Omega_1, \Omega_2 \subseteq \mathbb{R}$.

Definition 2.16. The family $\{E_\Omega \mid \Omega \subseteq \mathbb{R} \text{ is borel measurable}\}$ described in Fact 2.15 is called the *spectral projection valued measure* (s.p.v.m.) generated by Q .

Fact 2.17 (Remark before Theorem VIII.6 in [21]). Let $v \in \mathcal{H}$. Then the set function such that for every Borel set $\Omega \subset \mathbb{R}$ assigns the value $\langle E_\Omega v | v \rangle$ is a Borel measure. In the case when $\Omega = (-\infty, \lambda)$, this measure is denoted $\langle E_\lambda v | v \rangle$.

Fact 2.18 (Spectral Theorem-Integral Decomposition form, Theorem VIII.6 in [21]). Let Q be a closed unbounded self-adjoint operator on H and let h be a (possibly unbounded) Borel measurable function on \mathbb{R} . Then the (possibly unbounded) operator $h(Q)$ such that for every $v \in H$

$$\langle \pi(h)v | v \rangle := \int_{-\infty}^{\infty} h(l) d\langle E_\lambda v | v \rangle,$$

whenever $v \in D(\pi(h))$, with

$$D(\pi(h)) := \{v \in H \mid \int_{-\infty}^{\infty} |h(l)|^2 d\langle E_\lambda v | v \rangle < \infty\},$$

is such that $\pi(h)$ satisfies properties 1-4 of Theorem 2.13 and if h is a bounded borel measurable function on \mathbb{R} , then $\pi(h)$ is exactly the operator described in Theorem 2.13.

Definition 2.19. The *essential spectrum* of a closed unbounded self adoint operator Q ($\sigma_e(Q)$) is the set of complex values λ such that for every bounded operator S on H and every compact operator K on H , we have that $(Q - \lambda I)S \neq I + K$

Fact 2.20. Let Q be a closed unbounded self adjoint operator on H . Then $\sigma_e(Q) \subseteq \sigma(Q) \subseteq \mathbb{R}$.

Proof. Clear by definition of $\sigma(Q)$. □

Theorem 2.21. Let Q be a closed unbounded self adjoint operator. Then, for every $\lambda \in \mathbb{R}$, the following conditions are equivalent:

i $\lambda \in \sigma_e(Q)$

ii For every $\epsilon > 0$, $\dim(E_{(\lambda-\epsilon, \lambda+\epsilon)}H) = \infty$

Proof. (i) \Rightarrow (ii) Asume that there exists $\epsilon > 0$ such that $E_{(\lambda-\epsilon, \lambda+\epsilon)}H$ finite dimensional. Let

$$h(x) = \frac{1 - \chi_{(\lambda-\epsilon, \lambda+\epsilon)}(x)}{x - \lambda}.$$

Then h is a bounded borel measurable function on \mathbb{R} . By Fact 2.13 (functional calculus), we have that,

$$f(Q)(Q - \lambda I) = (Q - \lambda I)f(Q) = I - \chi_{(\lambda-\epsilon, \lambda+\epsilon)}(Q) = I - E_{(\lambda-\epsilon, \lambda+\epsilon)}H$$

Since $E_{(\lambda-\epsilon, \lambda+\epsilon)}(Q)$ is finite dimensional, it is compact and $\lambda \notin \sigma_e(Q)$

(ii) \Rightarrow (i) Suppose that $\lambda \notin \sigma_e(Q)$. Then there are a bounded operator S and a compact operator K such that,

$$(1) \quad S(Q - \lambda I) = (Q - \lambda I)S = I + K$$

Suppose that for some $v \in H$, $(Q - \lambda I)v = 0$. Then $(I - K)v = 0$ and, therefore, $Kv = -v$. Since K is compact, this implies that $\text{Ker}(Q - \lambda I)$ is finite dimensional By Hypotesis, for all $\epsilon > 0$, $\chi_{(\lambda-\epsilon, \lambda+\epsilon)}(Q)$ is infinite dimensional and contains $\text{ker}(Q - \lambda I)$ which is finite dimensional. So, for every $\epsilon > 0$ there exists $v_\epsilon \in \chi_{(\lambda-\epsilon, \lambda+\epsilon)}(Q)$ such that $\|v_\epsilon\| = 1$ and $d(v_\epsilon, \text{ker}(Q - \lambda I)) = 1$ By Theorem 2.18

$$\begin{aligned} \|(Q - \lambda I)v_\epsilon\|^2 &= \langle (Q - \lambda I)^*(Q - \lambda I)\chi_{(\lambda-\epsilon, \lambda+\epsilon)}(Q)(v_\epsilon) | v_\epsilon \rangle = \\ &= \int_{\lambda-\epsilon}^{\lambda+\epsilon} |x - \lambda|^2 d\langle E_x v_\epsilon | v_\epsilon \rangle \leq \int_{\lambda-\epsilon}^{\lambda+\epsilon} |x - \lambda|^2 dx \leq \epsilon^2 \int_{\lambda-\epsilon}^{\lambda+\epsilon} dx \leq 2\epsilon^3 \end{aligned}$$

and hence $Qv_\epsilon - \lambda v_\epsilon \rightarrow 0$ when $\epsilon \rightarrow 0$. From (1) we get:

$$v_\epsilon + kv_\epsilon = S(Qv_\epsilon - \lambda v_\epsilon) \rightarrow 0 \text{ when } \epsilon \rightarrow 0.$$

By compactness of k , there exists a sequence $(v_n) \subseteq \{v_\epsilon \mid \epsilon > 0\}$ such that $kv_n \rightarrow v$ when $n \rightarrow \infty$ for some $v \in H$. It follows that $v_n \rightarrow -v$ and, since $\|v_n\| = 1$, we get $\|v\| = 1$. Since $Q(v_n) - \lambda v_n \rightarrow 0$ when $n \rightarrow \infty$, we get $Qv = \lambda v$, and hence:

$$\|v_n - v\| \geq d(v_n, \text{ker}(Q - \lambda I)) = 1,$$

which is a contradiction. □

Definition 2.22. Let Q be a closed unbounded self adjoint operator on H . The *discrete spectrum* of Q is the set:

$$\sigma_d(Q) := \sigma(Q) \setminus \sigma_e(Q)$$

Definition 2.23. Let Q_1 and Q_2 be closed unbounded self adjoint operators defined on Hilbert spaces H_1 and H_2 respectively. Then (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) are said to be *spectrally equivalent* ($Q_1 \sim_\sigma Q_2$) if both of the following conditions hold:

- (1) $\sigma(Q_1) = \sigma(Q_2)$.
- (2) $\sigma_e(Q_1) = \sigma_e(Q_2)$.
- (3) $\dim\{x \in H_1 \mid Q_1 x = \lambda x\} = \dim\{x \in H_2 \mid Q_2 x = \lambda x\}$ for $\lambda \in \sigma(Q_1) \setminus \sigma_e(Q_1)$.

Theorem 2.24 (Classical Weyl theorem, Example 3 of Section XIII.4 in [21]). *If Q is a (possibly unbounded) self-adjoint operator and K is a compact operator on H . Then $\sigma_e(Q) = \sigma_e(Q + K)$.*

Lemma 2.25 (See Lemma II.4.3 in [13]). *Suppose X is a metric space. Let $\{\xi_k \mid k \geq 1\}$ and $\{\zeta_k \mid k \geq 1\}$ be two countable dense subsets of X such that each isolated point of X is repeated the same number of times in each sequence. Then, given $\epsilon > 0$ there is a permutation π of \mathbb{Z}_+ such that $\text{dist}(\xi_k, \zeta_{\pi(k)}) < \epsilon$ for all $k \geq 1$ and*

$$\lim_{k \rightarrow \infty} \text{dist}(\xi_k, \zeta_{\pi(k)}) = 0$$

Remark 2.26. In [13], Lemma II.4.3 states the same thing for X being a compact metric space. However, as we will see, proof works even in the case of a general metric space.

Proof of Lemma 2.25. Without loss of generality, we assume that X has no isolated point of X is repeated a finite number of times. Let $\epsilon > 0$ be given. Let k be a positive integer and let us assume by induction that $\pi(i)$ and $\pi^{-1}(i)$ are already defined for $i = 1, \dots, k-1$.

If $\pi(k)$ is not yet defined, we take the least $l \neq \{\pi(1), \dots, \pi(k-1)\}$ such that

$$|\xi_k - \zeta_l| < \frac{\epsilon}{2^k}$$

This is possible since $\{\zeta_n \mid n \geq 1, n \neq \{\pi(1), \dots, \pi(k-1)\}\}$ is still dense in X . Similarly, if $\pi^{-1}(k)$ is not yet defined, we take the least $l \neq \{\pi^{-1}(1), \dots, \pi^{-1}(k-1)\}$ such that

$$|\xi_l - \zeta_k| < \frac{\epsilon}{2^k}$$

This is possible since $\{\xi_n \mid n \geq 1, n \neq \{\pi^{-1}(1), \dots, \pi^{-1}(k-1)\}\}$ is still dense in X .

Defining π this way, every $k \in \mathbb{Z}_+$ will be eventually included in the domain and in the range of π once and only once. So it defines a permutation of \mathbb{Z}_+ with the desired properties. \square

Theorem 2.27 (Weyl-Von Neumann-Berg, Corollary 2 in [10]). *Let Q be a not necessarily bounded self adjoint operator on a separable Hilbert space H . Then for every $\epsilon > 0$ there exists a diagonal operator D and a compact operator K on H such that $\|K\| < \epsilon$ and $Q = D + K$.*

Definition 2.28. Two unbounded closed self adjoint operators Q_1 and Q_2 on a separable Hilbert spaces H_1 and H_2 are said to be *approximately unitarily equivalent* if there exists a sequence of unitary operators $(U_n)_{n < \omega}$ from H_1 to H_2 such that for every $n \in \mathbb{Z}_+$, $Q_2 - U_n Q_1 U_n^*$ is compact and for all $\epsilon > 0$, there is n_ϵ such that for every $n \geq n_\epsilon$, $\|Q_2 - U_n Q_1 U_n^*\| < \epsilon$.

Theorem 2.29 (See II.4.4 in [13]). *Suppose Q_1 and Q_2 are unbounded closed self adjoint operators on a separable Hilbert space H . Then Q_1 and Q_2 are approximately unitarily equivalent if and only if $Q_1 \sim_\sigma Q_2$.*

Remark 2.30. As in Lemma 2.25, we give a proof which is very similar to the one presented in Lemma II.4.3 in [13].

Proof of Theorem 2.29. \Rightarrow : Suppose Q_1 and Q_2 are approximately unitarily equivalent, and let $(U_n)_{n < \omega}$ be unitary operators from H to H such that for every $n \in \mathbb{Z}_+$, $Q_2 - U_n Q_1 U_n^*$ is compact and for all $\epsilon > 0$, there is n_ϵ such

that for every $n \geq n_\epsilon$, $\|Q_2 - U_n Q_1 U_n^*\| < \epsilon$. It is clear that $\sigma(Q_1) = \sigma(Q_2)$ and if h is a bounded Borel function on \mathbb{R} then

$$f(Q_2 - U_n Q_1 U_n^*) = f(Q_2) - U_n f(Q_1) U_n^*$$

is compact and for all $\epsilon > 0$, there is n_ϵ such that for every $n \geq n_\epsilon$,

$$\|f(Q_2) - U_n f(Q_1) U_n^*\| < \epsilon.$$

In particular, if $f = \chi_{\{x\}}$ is the characteristic function of an isolated point $\sigma(Q_1)$, then for every $n \in \mathbb{Z}_+$,

$$E_{Q_2}(\{x\}) - U_n E_{Q_1}(\{x\}) U_n^* = \chi_{\{x\}}(Q_2) - U_n \chi_{\{x\}}(Q_1) U_n^*$$

is compact and for all $\epsilon > 0$, there is n_ϵ such that for every $n \geq n_\epsilon$,

$$\|E_{Q_2}(\{x\}) - U_n E_{Q_1}(\{x\}) U_n^*\| = \|\chi_{\{x\}}(Q_2) - U_n \chi_{\{x\}}(Q_1) U_n^*\| < \epsilon$$

So, the eigenspace of every isolated point in $\sigma(Q_1) = \sigma(Q_2)$ has the same dimension, $\sigma_e(Q_1) = \sigma_e(Q_2)$ and $\sigma_d(Q_1) = \sigma_d(Q_2)$.

⇐: Let us suppose at first that Q_1 and Q_2 are diagonal, and let $Q_1 = \text{diag}(\xi_k)$ with respect to a basis v_k and $Q_2 = \text{diag}(\zeta_k)$ with respect to a basis w_k . Then (ξ_k) and (ζ_k) are dense in $\sigma(Q_1) = \sigma(Q_2)$ and if λ is an isolated point in $\sigma(Q_1) = \sigma(Q_2)$, then λ is repeated the same number of times (the dimension of its corresponding eigenspace). Therefore $\sigma_e(Q_1) = \sigma_e(Q_2)$.

Let $X := \sigma_e(Q_1) = \sigma_e(Q_2)$. Given $\epsilon > 0$, by Lemma 2.25, there is a permutation π of \mathbb{Z}_+ such that $|\xi_k, \zeta_{\pi(k)}| < \epsilon$ for all k and

$$\lim_{n \rightarrow \infty} |\xi_k, \zeta_{\pi(k)}| = 0$$

Then the unitary operator given by $U v_k := w_{\pi(k)}$ is such that the operator

$$Q_1 - U Q_2 U^* = \text{diag}(\xi_k - \zeta_{\pi(k)})$$

is compact and has norm less than ϵ .

For the more general case, Q_1 can be decomposed as $Q = Q_1^d \oplus Q_1'$ such that $\sigma(Q_1^d) = \sigma_d(Q_1^d) = \sigma_d(Q)$, the dimensions of the eigenspaces of elements in Q_1^d are the same as the dimensions of the eigenspaces of values in $\sigma_d(Q)$ and $\sigma(Q') = \sigma_e(Q)$. The something happens for Q_2 and Q_1^d and Q_2^d coincide. Because of this we can assume that $\sigma_d(Q) = \emptyset$. Now, given $\epsilon > 0$ there are diagonal operators D_1 and D_2 and compact operators K_1 and K_2 such that $Q_1 = D_1 + K_1$, $Q_2 = D_2 + K_2$ and $\|K_1\|, \|K_2\| < \epsilon$. By Weyl's theorem (Theorem 2.24) $\sigma(D_1) = \sigma_e(Q_1) = \sigma_e(Q_2) = \sigma(D_2)$. By the diagonal operators case, this implies that Q_1 and Q_2 are approximately unitarily equivalent. \square

Definition 2.31. Let Q be a closed unbounded selfadjoint operator on a Hilbert space H . For $\lambda \in \sigma_d(Q)$, let n_λ be the dimension of the eigenspace corresponding to λ . We define the *discrete part* of H in the following way:

$$H_d := \bigoplus_{\lambda \in \sigma_d(Q)} \mathbb{C}^{n_\lambda}$$

In the same way, we define $Q_d := Q \upharpoonright H_d$

Fact 2.32. $H_d \subseteq H$

Definition 2.33. Let Q be a closed unbounded selfadjoint operator on a Hilbert space H . We define the *essential part* of H in the following way:

$$H_e := H_d^\perp$$

In the same way, we define $Q_e := Q \upharpoonright H_e$

Definition 2.34. Given $G \subseteq H$ and $v \in H$, we denote by:

- (1) H_G , the Hilbert subspace of H generated by the elements $f(Q)v$, where $v \in G$, f is a bounded Borel function on \mathbb{R} and $v \in D(f(Q))$.
- (2) $Q_G := Q \upharpoonright H_G$.
- (3) H_v , the space H_G when $G = \{v\}$ for some vector $v \in H$
- (4) $Q_v := Q_G$ when $G = \{v\}$.
- (5) H_G^\perp , the orthogonal complement of H_G
- (6) P_G , the projection over H_G .
- (7) P_{G^\perp} , the projection over H_G^\perp .

Definition 2.35. Given $G \subseteq H$ and $v \in H$, we denote by $(H_G)_d$ and $(H_G)_e$ the projections of H_G on H_d and H_e respectively.

Definition 2.36. Let $v \in H$, the *spectral measure defined by v* (denoted by μ_v) is the finite borel measure that to any borel set $\Omega \subseteq \mathbb{R}$ assigns the (complex) number,

$$\mu_v(\Omega) := \langle \chi_\Omega(Q)v \mid v \rangle$$

Lemma 2.37 (Lemma XII.3.1 in [15]). *For $v \in H$, the space $H_v \simeq L^2(\mathbb{R}, \mu_v)$.*

Lemma 2.38 (Lemma XII.3.2 in [15]). *There is a set $G \subseteq H$ such that $H = \bigoplus_{v \in G} H_v$.*

Corollary 2.39. *There is a set $G \subseteq H$ such that $H = H_d \oplus \bigoplus_{v \in G} H_v$*

3. A METRIC ABSTRACT ELEMENTARY DEFINED BY $(H; Q)$

In this section we define a *metric abstract elementary class* associated with a closed unbounded self-adjoint operator Q defined on a Hilbert space (see Definition 3.5). We will recall several notions related with metric abstract elementary classes that come from [17].

Definition 3.1. An \mathcal{L} -metric structure \mathcal{M} , for a fixed similarity type \mathcal{L} , consists of:

- A closed metric space (M, d)
- A family $(R^{\mathcal{M}})_{R \in \mathcal{L}}$ of continuous functions from M^{n_R} into \mathbb{R} , where n_R is the arity of R .
- An indexed family $(F^{\mathcal{M}})_{F \in \mathcal{L}}$ of continuous functions on powers of M .
- An indexed family $(c^{\mathcal{M}})_{c \in \mathcal{L}}$ of distinguished elements of M .

We write this structure as

$$\mathcal{M} = (M, d, (R^{\mathcal{M}})_{R \in \mathcal{L}}, (F^{\mathcal{M}})_{F \in \mathcal{L}}, (c^{\mathcal{M}})_{c \in \mathcal{L}}).$$

If \mathcal{M} is a metric structure, $\text{dens}(\mathcal{M})$ denotes the smallest cardinal of a dense subset of M .

Definition 3.2. Let $\mathcal{L} = (0, -, i, +, (I_r)_{r \in \mathbb{Q}}, \|\cdot\|, \Gamma_Q)$. A *Hilbert space operator structure* for \mathcal{L} is a metric structure of only one sort:

$$(H, 0, +, i, (I_r)_{r \in \mathbb{Q}}, \|\cdot\|, \Gamma_Q)$$

where

- H is a Hilbert space
- Q is a closed (unbounded) selfadjoint operator on H
- 0 is the zero vector in H
- $+$: $H \times H \rightarrow H$ is the usual sum of vectors in H
- i : $H \rightarrow H$ is the function that to any vector $v \in H$ assigns the vector iv where $i^2 = -1$
- I_r : $H \rightarrow H$ is the function that sends every vector $v \in H$ to rv , where $r \in \mathbb{Q}$
- $\|\cdot\|$: $H \rightarrow \mathbb{R}$ is the norm function
- Γ_Q : $H \times H \rightarrow \mathbb{R}$ is the function that to any $v, w \in H$ assigns the number $\Gamma_Q(v, w)$, which is the distance of (v, w) to the graph of Q . Since Q is closed, $\Gamma_Q(v, w) = 0$ if and only if (v, w) belongs to the graph of Q .

Briefly, the structure will be referred to either as (H, Γ_Q) . (H, Γ_Q) is a metric structure for the similarity type

Lemma 3.3. *Let Q_1 and Q_2 be closed unbounded self adjoint operators defined on Hilbert spaces H_1 and H_2 respectively. An isomorphism $U : (H_1, \Gamma_{Q_1}) \rightarrow (H_2, \Gamma_{Q_2})$ is a unitary operator of $U : H_1 \rightarrow H_2$ such that $UD(Q_1) = D(Q_2)$ and $UQ_1v = Q_2Uv$ for every $v \in D(Q_1)$.*

Proof. \Rightarrow : Suppose U is an isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) . It is clear that U must be a linear operator. Also, we have that for every $u, v \in \mathcal{H}$ we must have that $\langle Uu | Uv \rangle = \langle u | v \rangle$ by definition of automorphism. Therefore U must be an isometry and, therefore it must be unitary.

On the other hand, since U is an isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) , for every $(v, w) \in H \times H$ we have that $\Gamma_{Q_1}(v, w) = \Gamma_{Q_2}(Uv, Uw)$. Therefore, $\Gamma_{Q_1}(v, w) = 0$ if and only if $\Gamma_{Q_2}(Uv, Uw) = 0$. So, for every $v \in D(Q_1)$, $UQ_1v = Q_2Uv$.

\Leftarrow : Let $U : H_1 \rightarrow H_2$ be a unitary operator such that $UD(Q_1) = D(Q_2)$ and $UQ_1v = Q_2Uv$ for every $v \in D(Q_1)$. It remains to show that for every $(v, w) \in H \times H$, $\Gamma_{Q_1}(v, w) = \Gamma_{Q_2}(Uv, Uw)$. Let $(v, w) \in H \times H$ be any pair of vectors. There exists a sequence of pairs $(v_n, w_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, $v_n \in D(Q_1)$, $w_n = Q_1v_n$ and $\Gamma_{Q_1}(v, w) = \lim_{n \rightarrow \infty} d[(v, w); (v_n, w_n)]$.

By hypothesis, U is an isometry, and maps the graph of Q_1 into the graph of Q_2 ; so for all $n \in \mathbb{N}$, $Uv_n \in D(Q_2)$ and $Uw_n = Q_2Uv_n$. We have that

$$\lim_{n \rightarrow \infty} d[(Uv, Uw); (Uv_n, Uw_n)] = \lim_{n \rightarrow \infty} d[(v, w); (v_n, w_n)] = \Gamma_{Q_1}(v, w).$$

So $\Gamma_{Q_2}(Uv, Uw) \leq \Gamma_{Q_1}(v, w)$. Repeating the argument for U^{-1} , we get $\Gamma_{Q_1}(v, w) \leq \Gamma_{Q_2}(Uv, Uw)$.

□

Fact 3.4. Let (H_1, Γ_{Q_1}) , (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) such that $Q_1 \sim_\sigma Q_2$. If (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) are isomorphic, then $Q_1 \sim_\sigma Q_3$.

Definition 3.5. A *Metric Abstract Elementary Class* (MAEC), on a fixed similarity type $\mathcal{L}(\mathcal{K})$, is a class \mathcal{K} of $\mathcal{L}(\mathcal{K})$ -metric structures provided with a partial order $\prec_{\mathcal{K}}$ such that:

- (1) Closure under isomorphism:
 - (a) For every $\mathcal{M} \in \mathcal{K}$ and every $\mathcal{L}(\mathcal{K})$ -structure \mathcal{N} , if $\mathcal{M} \simeq \mathcal{N}$ then $\mathcal{N} \in \mathcal{K}$.
 - (b) Let $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{K}$ and $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}$ be such that there exists $f_l : \mathcal{N}_l \simeq \mathcal{M}_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$. Then $\mathcal{N}_1 \prec_{\mathcal{K}} \mathcal{N}_2$ implies that $\mathcal{M}_1 \prec_{\mathcal{K}} \mathcal{M}_2$.
- (2) For all $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ if $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$.
- (3) Let \mathcal{M}, \mathcal{N} and \mathcal{M}^* be $\mathcal{L}(\mathcal{K})$ -structures. If $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}^*$ and $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}^*$, then $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$.
- (4) Downward Löwenheim-Skolem: There exists a cardinal $LS(\mathcal{K}) \geq \aleph_0 + |\mathcal{L}(\mathcal{K})|$ such that for every $\mathcal{M} \in \mathcal{K}$ and for every $A \subseteq M$ there exists $\mathcal{N} \in \mathcal{K}$ such that $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}$, $N \supseteq A$ and $\text{dens}(N) \leq |A| + LS(\mathcal{K})$.
- (5) Tarski-Vaught chain:
 - (a) For every cardinal μ and every $\mathcal{N} \in \mathcal{K}$, if $\{\mathcal{M}_i \prec_{\mathcal{K}} \mathcal{N} \mid i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$ -increasing and continuous (i.e. $i < j \Rightarrow \mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_j$) then $\overline{\bigcup_{i < \mu} \mathcal{M}_i} \in \mathcal{K}$ and $\overline{\bigcup_{i < \mu} \mathcal{M}_i} \prec_{\mathcal{K}} \mathcal{N}$.
 - (b) For every μ , if $\{\mathcal{M}_i \mid i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$ -increasing (i.e. $i < j \Rightarrow \mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_j$) and continuous then $\overline{\bigcup_{i < \mu} \mathcal{M}_i} \in \mathcal{K}$ and for every $j < \mu$, $\mathcal{M}_j \prec_{\mathcal{K}} \overline{\bigcup_{i < \mu} \mathcal{M}_i}$.

Here, $\overline{\bigcup_{i < \mu} \mathcal{M}_i}$ denotes the completion of $\bigcup_{i < \mu} \mathcal{M}_i$.

Definition 3.6. Let (H, Γ_Q) be a structure as described in Definition 3.2. Let \mathcal{L} the similarity type of (H, Γ_Q) . We define $\mathcal{K}_{(H, \Gamma_Q)}$ to be the following class:

$$\mathcal{K}_{(H, \Gamma_Q)} := \{(H', Q') \mid (H', Q') \text{ is an } \mathcal{L} \text{ Hilbert space operator structure and } Q' \sim_{\sigma} Q\}$$

We define the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H, \Gamma_Q)}$ by:

$$(H_1, \Gamma_{Q_1}) \prec_{\mathcal{K}} (H_2, \Gamma_{Q_2}) \text{ if and only if } H_1 \subseteq H_2 \text{ and } Q_1 \subseteq Q_2$$

Theorem 3.7. *The class $\mathcal{K}_{(H, \Gamma_Q)}$ is a MAEC.*

Proof. (1) Closure under isomorphism:

- (a) Clear by Lemma 3.3.
- (b) Clear.
- (2) Clear.
- (3) Clear.
- (4) $LS(\mathcal{K}) \leq 2^{2^{\aleph_0}}$. We first prove following claim:

Claim. If $(H', Q') \in \mathcal{K}_{(H, \Gamma_Q)}$, there is a $(H'', Q'') \prec (H', Q')$ such that $(H'', Q'') \in \mathcal{K}$ and $|H''| \leq 2^{2^{\aleph_0}}$.

Proof. By Corollary 2.39, there is a set $G' \subseteq H'$ such that $H' = H_d \oplus \bigoplus_{v \in G'} H'_v$. Since there are at most $2^{2^{\aleph_0}}$ many Borel measures, there is a $G'' \subseteq G'$ such that $|G''| \leq 2^{2^{\aleph_0}}$ and for every $v \in G'$ there is a $w \in G''$ such that $\mu_v = \mu_w$. Take

$$H'' = H_d \oplus \bigoplus_{v \in G''} H'_v$$

and

$$Q'' := Q' \upharpoonright H''$$

Then $(H'', Q'') \in \mathcal{K}_{(H, \Gamma_Q)}$, $(H'', Q'') \prec (H', Q')$ and $|H''| \leq 2^{2^{\aleph_0}}$. \square

Now, let $(H', Q') \in \mathcal{K}$ and $A \subseteq H'$. Let G' be as in Corollary 2.39 and let (H'', Q'') be as in previous Claim. Since $A \subseteq H_d \oplus \bigoplus_{v \in G''} H'_v$, there is a $G_A \subseteq G''$ tal que $|G_A| \leq |A| \aleph_0$ such that $A \subseteq H_d \oplus \bigoplus_{v \in G_A} H'_v$.

Let

$$\hat{H} := H_d \oplus \bigoplus_{v \in G_A \cup G''} H'_v$$

and

$$Q'' := Q' \upharpoonright \hat{H}$$

Then $(\hat{H}, \hat{Q}) \in \mathcal{K}_{(H, \Gamma_Q)}$, $(\hat{H}, \hat{Q}) \prec (H', Q')$, $A \subseteq \hat{A}$ and $|\hat{H}| \leq |A| + 2^{2^{\aleph_0}}$.

(5) Tarski-Vaught chain:

- (a) Suppose κ is a regular cardinal and $(\hat{H}, \hat{Q}) \in \mathcal{K}_{(H, \Gamma_Q)}$. Let $(H_i, \Gamma_{Q_i})_{i < \kappa}$ a $\prec_{\mathcal{K}}$ increasing sequence such that $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \hat{Q})$ for all $i < \kappa$. Then, for all $i < \kappa$ $(H_{i+1}, Q_{i+1}) = (H_i, \Gamma_{Q_i}) \oplus (H'_i, Q'_i)$, where H'_i is a Hilbert space and Q'_i is a (possibly unbounded) closed selfadoint operator such that $\sigma_d(Q'_i) = \emptyset$ and $\sigma_e(Q'_i) \subseteq \sigma_e(\hat{Q})$. Then $\overline{\bigcup_{i < \kappa} (H_i, \Gamma_{Q_i})} = H_0 \oplus \bigoplus_{i < \kappa} (H'_i, Q'_i)$. Since $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \hat{Q})$, $\overline{\bigcup_{i < \kappa} (H_i, \Gamma_{Q_i})} \prec_{\mathcal{K}} (\hat{H}, \hat{Q})$.
- (b) Clear from previous item.

□

Remark 3.8. From now on, the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H, \Gamma_Q)}$ will be denoted as \prec .

Definition 3.9. Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be a MAEC and let $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ be two structures. An embedding $f : \mathcal{M} \rightarrow \mathcal{N}$ such that $f(\mathcal{M}) \prec_{\mathcal{K}} \mathcal{N}$ is called a \mathcal{K} -embedding.

Definition 3.10. A MAEC \mathcal{K} has the *Joint Embedding Property* (JEP) if for any $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}$ there are $\mathcal{N} \in \mathcal{K}$ and a \mathcal{K} -embeddings $f : \mathcal{M}_1 \rightarrow \mathcal{N}$ and $g : \mathcal{M}_2 \rightarrow \mathcal{N}$.

Theorem 3.11. $\mathcal{K}_{(H, \Gamma_Q)}$ has the JEP.

Proof. Let $(H_1, \Gamma_{Q_1}), (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H, \Gamma_Q)}$. Wiothout loss of generality, we can assume that $H_1 \cap H_2 = \emptyset$. By Corollary 2.39, there are sets $G_1 \subseteq H_1$ and $G_2 \subseteq H_2$ such that $H_1 = H_d \oplus \bigoplus_{v \in G_1} (H_1)_v$ and $H_2 = H_d \oplus \bigoplus_{v \in G_2} (H_2)_v$.

Let

$$\hat{H} = H_d \oplus \bigoplus_{v \in G_1} L^2(\mathbb{R}, \mu_v) \oplus \bigoplus_{v \in G_2} L^2(\mathbb{R}, \mu_v)$$

By the Spectral Theorem-Multiplication Form and Lemma 2.37 for every $v \in G_1 \cup G_2$, there is a Borel function f_v and an isomorphism $U_v : H_v \rightarrow L^2(\mathbb{R}, \mu_v)$ such that $(H_v, Q \upharpoonright H_v) \simeq (L^2(\mathbb{R}, \mu_v), M_{f_v})$, where M_{f_v} is the multiplication by f_v in $L^2(\mathbb{R}, \mu_v)$. If

$$\hat{Q} := (Q_1 \upharpoonright H_d) \oplus \left(\bigoplus_{v \in G_1} M_{f_v} \right) \oplus \left(\bigoplus_{v \in G_2} M_{f_v} \right)$$

then, $id_{H_d} \oplus \bigoplus_{v \in G_1} U_v$ and $id_{H_d} \oplus \bigoplus_{v \in G_2} U_v$ are respective $\mathcal{K}_{(H, \Gamma_Q)}$ -embeddings from (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) to (\hat{H}, \hat{Q}) . □

Definition 3.12. A MAEC \mathcal{K} has the *Amalgamation Property* (AP) if for any $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2 \in \mathcal{K}$ such that $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}_1$ and $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}_2$, there are $\mathcal{M}' \in \mathcal{K}$ and a \mathcal{K} -embeddings $f : \mathcal{N}_1 \rightarrow \mathcal{M}'$ and $g : \mathcal{N}_2 \rightarrow \mathcal{M}'$ such that $f(\mathcal{N}_1), g(\mathcal{N}_2) \prec_{\mathcal{K}} \mathcal{M}'$. and $f \upharpoonright \mathcal{M} = g \upharpoonright \mathcal{M}$.

Theorem 3.13. $\mathcal{K}_{(H, \Gamma_Q)}$ has the AP.

Proof. Let (H_1, Γ_{Q_1}) , (H_2, Γ_{Q_2}) and $(H_3, \Gamma_{Q_3}) \in \mathcal{K}_{(H, \Gamma_Q)}$ be such that $(H_1, \Gamma_{Q_1}) \prec (H_2, \Gamma_{Q_2})$ and $(H_1, \Gamma_{Q_1}) \prec (H_3, \Gamma_{Q_3})$. By Corollary 2.39, there are sets $G_1 \subseteq H_1$, $G_2 \subseteq H_2$ and $G_3 \subseteq H_3$ such that:

- $H_1 = H_d \oplus \bigoplus_{v \in G_1} (H_1)_v$
- $H_2 = H_d \oplus \bigoplus_{v \in G_1} (H_1)_v \oplus \bigoplus_{v \in G_2} (H_2)_v$
- $H_3 = H_d \oplus \bigoplus_{v \in G_1} (H_1)_v \oplus \bigoplus_{v \in G_3} (H_3)_v$

Let

$$H_4 := H_d \oplus \bigoplus_{v \in G_1} L^2(\mathbb{R}, \mu_v) \oplus \bigoplus_{v \in G_2} L^2(\mathbb{R}, \mu_v) \oplus \bigoplus_{v \in G_3} L^2(\mathbb{R}, \mu_v)$$

and

$$Q_4 := (Q_1 \upharpoonright H_d) \oplus \left(\bigoplus_{v \in G_1} M_{f_v} \right) \oplus \left(\bigoplus_{v \in G_2} M_{f_v} \right) \oplus \left(\bigoplus_{v \in G_3} M_{f_v} \right)$$

Then $(H_4, Q_4) \in \mathcal{K}_{(H, \Gamma_Q)}$ and $id_{H_d} \oplus \bigoplus_{v \in G_1} U_v \oplus \bigoplus_{v \in G_2} U_v, id_{H_d} \oplus \bigoplus_{v \in G_1} U_v \oplus \bigoplus_{v \in G_3} U_v$ are respective $\mathcal{K}_{(H, \Gamma_Q)}$ -embeddings from (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) to (H_4, Q_4) . \square

Remark 3.14. For (H_1, Γ_{Q_1}) , (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) as in Theorem 3.13, we denote by

$$(H_2, \Gamma_{Q_2}) \bigvee_{(H_1, \Gamma_{Q_1})} (H_3, \Gamma_{Q_3}) := (H_2 \vee_{H_2} H_3, Q_2 \vee_{Q_1} Q_3)$$

the *amalgamation* of (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) over (H_1, Γ_{Q_1}) as described in Theorem 3.13.

Definition 3.15. For $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}$, $A \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$ and $(a_i)_{i < \alpha} \subseteq \mathcal{M}_1$, $(b_i)_{i < \alpha} \subseteq \mathcal{M}_2$, we say that $(a_i)_{i < \alpha}$ and $(b_i)_{i < \alpha}$ have the same *Galois type* in \mathcal{M}_1 and \mathcal{M}_2 respectively, $(\text{gatp}_{\mathcal{M}_1}((a_i)_{i < \alpha}/A) = \text{gatp}_{\mathcal{M}_2}((b_i)_{i < \alpha}/A))$, if there are $\mathcal{N} \in \mathcal{K}$ and \mathcal{K} -embeddings $f : \mathcal{M}_1 \rightarrow \mathcal{N}$ and $g : \mathcal{M}_2 \rightarrow \mathcal{N}$ such that $(a_i) = g(b_i)$ for every $i < \alpha$ and $f \upharpoonright A \equiv g \upharpoonright A \equiv Id_A$, where Id_A is the identity on A .

Theorem 3.16. Let $v \in (H_1, \Gamma_{Q_1})$, $w \in (H_2, \Gamma_{Q_2})$ and $G \subseteq H_1 \cap H_2$ such that $(H_G, \Gamma_{Q_G}) \in \mathcal{K}_{(H, \Gamma_Q)}$, $(H_G, \Gamma_{Q_G}) \prec (H_1, \Gamma_{Q_1})$, $(H_G, \Gamma_{Q_G}) \prec (H_2, \Gamma_{Q_2})$, $G \subseteq H_1 \cap H_2$. Then $\text{gatp}_{(H_1, \Gamma_{Q_1})}(v/G) = \text{gatp}_{(H_2, \Gamma_{Q_2})}(w/G)$ if and only if

$$P_G v = P_G w$$

and

$$\mu_{P_{G^\perp} v} = \mu_{P_{G^\perp} w}.$$

Proof. \Rightarrow : Suppose $\text{gatp}_{(H_1, \Gamma_{Q_1})}(v/G) = \text{gatp}_{(H_2, \Gamma_{Q_2})}(w/G)$ and let $v' := P_{G^\perp} v$ and $w' := P_{G^\perp} w$. Then, by Definition 3.15, there exists $(H_3, \Gamma_{Q_3}) \in \mathcal{K}_{(H, \Gamma_Q)}$ and $\mathcal{K}_{(H, \Gamma_Q)}$ -embeddings $U_1 : (H_1, \Gamma_{Q_1}) \rightarrow (H_3, \Gamma_{Q_3})$ and $U_2 : (H_2, \Gamma_{Q_2}) \rightarrow (H_3, \Gamma_{Q_3})$ such that $U_1 v = U_2 w$ and $U_1 \upharpoonright G \equiv U_2 \upharpoonright G \equiv Id_G$, where Id_G is the identity on G . Since $v = P_G v + P_{G^\perp} v$, $w = P_G w + P_{G^\perp} w$ and $U_1 \upharpoonright G \equiv U_2 \upharpoonright G \equiv Id_G$, we have that $U_1 P_G v = P_G v$ and $U_2 P_G w = P_G w$. Since U_1 and U_2 are embeddings, $\mu_{v'} = \mu_{U_1 v'} = \mu_{U_2 w'} = \mu_{w'}$.

\Leftarrow : Let $v' := P_{G^\perp} v$ and $w' := P_{G^\perp} w$. Suppose $\mu_{v'} = \mu_{w'}$, then $\mu_{v'_e} = \mu_{w'_e}$ $L^2(\mathbb{R}, \mu_{v'_e}) = L^2(\mathbb{R}, \mu_{w'_e})$. Let $\mu := \mu_{v'_e} = \mu_{w'_e}$. Also, let

$$\hat{H} := (H_1 \vee_{H_G} H_2) \oplus L^2(\mathbb{R}, \mu)$$

and let

$$\hat{Q} := (Q_1 \vee_{Q_G} Q_2) \oplus M_{f_\mu}$$

be as in the Spectral Theorem-Multiplication form. Let $U_1 : (H_1, \Gamma_{Q_1}) \rightarrow (\hat{H}, \hat{Q})$ be the $\mathcal{K}_{(H, \Gamma_Q)}$ -embedding acting on $H_{v'}^\perp \vee H_{w'}^\perp$ as in the AP, and acting on $H_{v'}$ as in Lemma 2.37. Define $U_2 : (H_2, \Gamma_{Q_2}) \rightarrow (\hat{H}, \hat{Q})$ in the same way. Then we have completed the conditions to show that $\text{gatp}_{(H_1, \Gamma_{Q_1})}(v/G) = \text{gatp}_{(H_2, \Gamma_{Q_2})}(w/G)$. \square

Definition 3.17. A MAEC \mathcal{K} is said to be *homogeneous* if whenever $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ and $(a_i)_{i < \alpha} \subseteq \mathcal{M}$, $(b_i)_{i < \alpha} \subseteq \mathcal{N}$ such that for all $n < \omega$ and $i_0, \dots, i_{n-1} < \alpha$

$$\text{gatp}_{\mathcal{M}}(a_{i_0}, \dots, a_{i_{n-1}}/\emptyset) = \text{gatp}_{\mathcal{N}}(b_{i_0}, \dots, b_{i_{n-1}}/\emptyset)$$

then we have that

$$\text{gatp}_{\mathcal{M}}((a_i)_{i < \alpha}/\emptyset) = \text{gatp}_{\mathcal{N}}((b_i)_{i < \alpha}/\emptyset),$$

Theorem 3.18. $\mathcal{K}_{(H, \Gamma_Q)}$ is an homogeneous MAEC.

Proof. Let $(H_1, \Gamma_{Q_1}), (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H, \Gamma_Q)}$ and $(v_i)_{i < \alpha} \subseteq H_1$, $(w_i)_{i < \alpha} \subseteq H_2$ be such that for all $n < \omega$ and $i_0, \dots, i_{n-1} < \alpha$

$$\text{gatp}_{(H_1, \Gamma_{Q_1})}(v_{i_0}, \dots, v_{i_{n-1}}/\emptyset) = \text{gatp}_{(H_2, \Gamma_{Q_2})}(w_{i_0}, \dots, w_{i_{n-1}}/\emptyset)$$

Without loss of generality, we can assume that for all $i < \alpha$ $v_i \in (H_1)_e$ and $w_i \in (H_2)_e$ and for every $i \neq j < \alpha$, $v_i \perp v_j$ and $w_i \perp w_j$. For $i < \alpha$, let $\mu_i := \mu_{v_i} = \mu_{w_i}$, which is possible by Theorem 3.16, since for all $i < \alpha$ $\text{gatp}_{(H_1, \Gamma_{Q_1})}(v_i/\emptyset) = \text{gatp}_{(H_2, \Gamma_{Q_2})}(w_i/\emptyset)$. Also, let

$$\hat{H} := (H_1 \vee_\emptyset H_2) \oplus \bigoplus_{i < \alpha} L^2(\mathbb{R}, \mu_i)$$

and

$$\hat{Q} := (Q_1 \vee_\emptyset Q_2) \oplus \bigoplus_{i < \alpha} M_{f_{\mu_i}}$$

be as in the Spectral Theorem-Multiplication form. Let $U_1 : (H_1, \Gamma_{Q_1}) \rightarrow (\hat{H}, \hat{Q})$ be the $\mathcal{K}_{(H, \Gamma_Q)}$ -embedding acting on $H_{(v_i)_{i < \alpha}}^\perp \vee H_{(w_i)_{i < \alpha}}^\perp$ as in the AP, and acting on $H_{(v_i)_{i < \alpha}}$ as in Lemma 2.37. Define $U_2 : (H_2, \Gamma_{Q_2}) \rightarrow (\hat{H}, \hat{Q})$ in the same way. Then we have completed the conditions to show that $\text{gatp}_{(H_1, \Gamma_{Q_1})}((v_i)_{i < \alpha}/\emptyset) = \text{gatp}_{(H_2, \Gamma_{Q_2})}((w_i)_{i < \alpha}/\emptyset)$. \square

Theorem 3.19 (Theorem 1.13 in [17]). *Let $(\mathcal{K}, \prec_{\mathcal{K}})$ a MAEC on a simlaity type Λ satisfying JEP, AP and homogeneity. Let $\kappa > |\Lambda| + LS(\mathcal{K})$, and then there is $\mathfrak{M} \in \mathcal{K}$ such that*

κ -universality: \mathfrak{M} is κ -universal, that is for all $\mathcal{M} \in \mathcal{K}$ such that $|\mathcal{M}| < \kappa$, there is a \mathcal{K} embedding $f : \mathcal{M} \rightarrow \mathfrak{M}$.

κ -homogeneity: \mathfrak{M} is κ -homogeneous, that is if $(a_i)_{i < \alpha}$, $(b_i)_{i < \alpha} \subseteq \mathfrak{M}$ are such that for all $n < \omega$ and $i_0, \dots, i_{n-1} < \alpha$

$$\text{gatp}_{\mathfrak{M}}(a_{i_0}, \dots, a_{i_{n-1}}/\emptyset) = \text{gatp}_{\mathfrak{M}}(b_{i_0}, \dots, b_{i_{n-1}}/\emptyset)$$

then there is an automorphism f of \mathfrak{M} such that $f(a_i) = b_i$ for all $i < \alpha$.

Definition 3.20. If in previous theorem, κ is a cardinal greater than the density of any structure in \mathcal{K} that we want to study, the structure \mathfrak{M} is called a *Monster Model*.

Remark 3.21. Let κ be as above, and let $\mathbb{M}(\mathbb{R})$ the set of all regular Borel measures on \mathbb{R} whose support is disjoint from $\sigma_p(Q)$. Then the structure $(\tilde{H}_\kappa, \tilde{Q}_\kappa)$ where

$$\tilde{H} = H_d \oplus \bigoplus_{\kappa} \left(\bigoplus_{\mu \in \mathbb{M}} L^2(\mathbb{R}, \mu) \right)$$

and

$$\tilde{Q} = (Q \upharpoonright H_d) \oplus \bigoplus_{\kappa} \left(\bigoplus_{\mu \in \mathbb{M}} M_{f_\mu} \right)$$

works as a monster model for $\mathcal{K}_{(H, \Gamma_Q)}$. This can be easily proven from the proofs of JEP, AP and homogeneity of $K_{(H, \Gamma_Q)}$.

4. DEFINABLE AND ALGEBRAIC CLOSURES

In this section we give a characterization of definable and algebraic closures. Definable closures are described in 4.2, while algebraic closures are characterized in 4.6.

Definition 4.1. Let \mathcal{K} be a MAEC with JEP and AP. Let \mathfrak{M} be the monster model in \mathcal{K} and let $A \subseteq \mathfrak{M}$ be a small subset. Then,

- (1) The *definable closure* of A is the set

$$dcl(A) := \{m \in \mathfrak{M} \mid Fm = m \text{ for all } F \text{ automorphism of } \mathfrak{M} \text{ that fixes } A \text{ pointwise}\}$$

- (2) The *algebraic closure* of A is the set

$$acl(A) := \{m \in \mathfrak{M} \mid \text{the orbit under all the automorphisms of } \mathfrak{M} \text{ that fix } A \text{ pointwise is compact}\}$$

Theorem 4.2. Let $G \subseteq \tilde{H}$. Then $dcl(G) = \tilde{H}_G$.

Proof. $dcl(G) \subseteq \tilde{H}_G$: Let $v \notin \tilde{H}_G$. Then $P_{G^\perp}v \neq 0$. Let $(H', Q') \in \mathcal{K}_{(H, \Gamma_Q)}$ be a small structure containing v . Let $(H'', Q'') \in \mathcal{K}_{(H, \Gamma_Q)}$ be a structure containing $H' \oplus L^2(\mathbb{R}, \mu_{P_{G^\perp}v_e})$. Let $w := P_G v + (1)_{\mu_{P_{G^\perp}v_e}} \in H''$. Then $gatp(v/G) = gatp(w/G)$, but $v \neq w$. Therefore $v \notin dcl(G)$.

$\tilde{H}_G \subseteq dcl(G)$: Let $v \in G$, let f be a bounded Borel function on \mathbb{R} , let $U \in Aut(\tilde{H}, \tilde{Q}/G)$ and let (H', Q') a small structure containing G . Then, by Lemma 3.3, $Uf(Q')v = f(Q')Uv = f(Q')v$, and $v \in dcl(G)$. \square

Lemma 4.3. Let $v \in \tilde{H}$. If v is an eigenvector corresponding to some $\lambda \in \sigma_d(N)$ then v is algebraic over \emptyset .

Proof. $\lambda \in \sigma_d(N)$ if and only if λ is isolated in $\sigma(N)$ with finite dimensional eigenspace \tilde{H}_λ . So any automorphism can only send \tilde{H}_λ onto \tilde{H}_λ and the orbit of v under such automorphism can only be compact. \square

Lemma 4.4. Let $v \in \tilde{H}$ be such that $v = \sum v_k$ where each v_k is an eigenvector for some $\lambda_k \in \sigma_d(N)$. Then v is algebraic over \emptyset .

Proof. Given that $\|v_k\| \rightarrow 0$ when $k \rightarrow \infty$, the orbit of v under all the automorphisms is a Hilbert cube which is compact. \square

Theorem 4.5. $\text{acl}(\emptyset) = H_d$

Proof. $\text{acl}(\emptyset) \subseteq H_d$ is a consequence of Lemma 4.4. For the converse, suppose $v \in \tilde{H}$ such that $v_e \neq 0$. Let κ be an uncountable small cardinal and let $G := \bigoplus_{\kappa} L^2(\mathbb{R}, \mu_{v_e})$. Any structure in $\mathcal{K}_{(H, \Gamma_Q)}$ containing G will have κ different realizations of $\text{gatp}(v/\emptyset)$. Therefore $v \notin \text{acl}(\emptyset)$. \square

Theorem 4.6. Let $G \subseteq \tilde{H}$. Then $\text{acl}(G)$ is closed Hilbert subspace generated by the union of $\text{dcl}(G)$ with $\text{acl}(\emptyset)$.

Proof. Let E be the space $\text{acl}(\emptyset) + \text{dcl}(G)$. We have that $\text{acl}(\emptyset) \subseteq \text{acl}(G)$ and $\text{dcl}(G) \subseteq \text{acl}(G)$ so $E \subseteq \text{acl}(G)$. If $v \notin E$, then $P_E^\perp v \neq 0$. Let κ be an uncountable small cardinal and let $G := \bigoplus_{\kappa} L^2(\mathbb{R}, \mu_{(P_E^\perp v)_e})$. Any structure in $\mathcal{K}_{(H, \Gamma_Q)}$ containing G will have κ different realizations of $\text{gatp}(v/G)$. Therefore, $v \notin \text{acl}(A)$. \square

5. PERTURBATIONS

In this section, perturbations of a structure $(H, \Gamma_Q) \in \mathcal{K}_{(H, \Gamma_Q)}$ are defined. Main results here are Theorem 5.2 and Theorem 5.5 that state that $\mathcal{K}_{(H, \Gamma_Q)}$ has the perturbation property and is a MAEC with perturbations respectively.

Definition 5.1. Let \mathcal{K} be a MAEC that satisfies the JEP, AP and homogeneity. Let \mathfrak{M} be its monster model. Then \mathcal{K} is said to have the *perturbation property* if whenever $A \subseteq \mathfrak{M}$ and $(b_i)_{i < \omega}$ is a convergent sequence with limit $b = \lim_{i \rightarrow \infty} b_i$ such that $\text{gatp}(b_i/A) = \text{gatp}(b_j/A)$ for all $i, j < \omega$, then $\text{gatp}(b/A) = \text{gatp}(b_i/A)$ for all $i < \omega$.

Theorem 5.2. $\mathcal{K}_{(H, \Gamma_Q)}$ has the perturbation property.

Proof. Let $G \subseteq \tilde{H}$ be small and $(v_i)_{i < \omega} \subseteq \tilde{H}$ a sequence such that $\lim_{i \rightarrow \infty} v_i = v$ and $\text{gatp}(v_i/G) = \text{gatp}(v_j/G)$ for all $i, j < \omega$. Then by Theorem 3.16, $P_G v_i = P_G v_j$ and $\text{gatp}(P_G^\perp v_i/\emptyset) = \text{gatp}(P_G^\perp v_j/\emptyset)$ for all $i, j < \omega$. If $\lim_{i \rightarrow \infty} v_i = v$, it is clear that $P_G v_i = P_G v$ for all $i < \omega$. So it is enough to prove the theorem for the case $G = \emptyset$.

Suppose $\lim_{i \rightarrow \infty} v_i = v$ and $\text{gatp}(v_i/\emptyset) = \text{gatp}(v_j/\emptyset)$ for all $i, j < \omega$. By Theorem 3.16, this means that $\mu_i = \mu_j$ for all $i, j < \omega$. Let $\mu := \mu_i$ and $E \subseteq \mathbb{R}$ be a Borel set. Then $\langle \chi_E(Q)v \mid v \rangle = \langle \chi_E(Q)(\lim_{i \rightarrow \infty} v_i \mid \lim_{i \rightarrow \infty} v_i) \rangle = \lim_{i \rightarrow \infty} \langle \chi_E(Q)v_i \mid v_i \rangle = \lim_{i \rightarrow \infty} \mu_i(E) = \lim_{i \rightarrow \infty} \mu(E) = \mu(E)$. Again by Theorem 3.16, $\text{gatp}(v_i/\emptyset) = \text{gatp}(v/\emptyset)$ for all $i < \omega$. \square

Definition 5.3. Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be a MAEC. A class $(\mathbb{F}_e)_{e \geq 0}$ collections of bijective mappings between members of \mathcal{K} is said to be a *system of perturbations* for $(\mathcal{K}, \prec_{\mathcal{K}})$ if

- (1) The \mathbb{F}_e are collections of bijective mappings between members of \mathcal{K} such that
- (2) $\mathbb{F}_\delta \subseteq \mathbb{F}_e$ if $\delta < e$, $\mathbb{F}_0 = \bigcup_{e > 0} \mathbb{F}_e$ and \mathbb{F}_0 is exactly the collection of real isomorphisms of structures in \mathcal{K} .
- (3) If $f : \mathcal{M} \rightarrow \mathcal{N}$ is in \mathbb{F}_e , then f is a e^ϵ -bi lipschitz mapping with respect to the metric i.e. $e^{-\epsilon}d(x, y) \leq d(f(x), f(y)) \leq e^\epsilon d(x, y)$ for all $x, y \in M$.
- (4) If $f \in \mathbb{F}_e$ then $f^{-1} \in \mathbb{F}_e$.
- (5) If $f \in \mathbb{F}_e$, $g \in \mathbb{F}_\delta$, and $\text{dom}(g) = \text{rng}(f)$ then $g \circ f \in \mathbb{F}_{e+\delta}$.

- (6) If $(f_i)_{i<\alpha}$ is an increasing chain of ϵ -isomorphisms, i.e. $f_i \in \mathbb{F}_\epsilon$, $f_i \mathcal{M}_i \rightarrow \mathcal{N}_i$, $\mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_{i+1}$, $\mathcal{N}_i \prec_{\mathcal{K}} \mathcal{N}_{i+1}$ and $f_i \subseteq f_{i+1}$ for every $i < \alpha$, then there is an ϵ -isomorphism $f : \bigcup_{i<\alpha} \mathcal{M}_i \rightarrow \bigcup_{i<\alpha} \mathcal{N}_i$ such that $f \upharpoonright \mathcal{M}_i = f_i$ for all $i < \omega$.

If $(\mathbb{F}_\epsilon)_{\epsilon \geq 0}$ is a system of perturbations for $(\mathcal{K}, \prec_{\mathcal{K}})$, then $(\mathcal{K}, \prec_{\mathcal{K}}, (\mathbb{F}_\epsilon)_{\epsilon \geq 0})$ is called a *MAEC with perturbations*.

Definition 5.4. Let $\epsilon > 0$. An ϵ -perturbation in $\mathcal{K}_{(H, \Gamma_Q)}$ is a unitary operator $U : H_1 \rightarrow H_2$ such that there are closed unbounded selfadjoint operators Q_1 and Q_2 defined on H_1 and H_2 respectively, such that

- (1) $(H_1, \Gamma_{Q_1}), (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H, \Gamma_Q)}$
- (2) $UD(Q_1) = D(Q_1)$
- (3) The operator $Q_1 - U^{-1}Q_2U$ can be extended to a bounded operator on H_1 with norm less than ϵ
- (4) The operator $Q_2 - UQ_1U^{-1}$ can be extended to a bounded operator on H_2 with norm less than ϵ

The class of all ϵ -perturbations in $\mathcal{K}_{(H, \Gamma_Q)}$ is denoted by $\mathbb{F}_\epsilon^{(H, \Gamma_Q)}$

Theorem 5.5. $(\mathcal{K}_{(H, \Gamma_Q)}, \prec_{\mathcal{K}_{(H, \Gamma_Q)}}, (\mathbb{F}_\epsilon^{(H, \Gamma_Q)})_{\epsilon \geq 0})$ is a MAEC with perturbations.

Proof. Items (1), (2), (3) and (4) are clear. (5) Comes from triangle inequality. Finally, For (6), recall from the Tarsky chain condition in Theorem 3.7 that $\bigcup_{i<\kappa} (H_i, \Gamma_{Q_i}) = H_0 \oplus_{i<\kappa} (H'_i, Q'_i)$. This with the fact that a direct sum of κ bounded operators with norm less than ϵ is still a bounded operator with norm less than ϵ . \square

6. STABILITY

Here, we prove superstability of the MAEC $\mathcal{K}_{(H, \Gamma_Q)}$ by counting types over sets and show that it is \aleph_0 -stable up to perturbations. This are the statements of Theorem 6.8 and Theorem 6.10 respectively.

Theorem 6.1. Let $v, w \in \tilde{H}$. Then \tilde{H}_v is isometrically isomorphic to a Hilbert subspace of \tilde{H}_w if and only if $\mu_v \ll \mu_w$.

Proof. By Radon Nikodim Theorem, if $\mu_u \ll \mu_v$ then \tilde{H}_v is isometrically equivalent to a Hilbert subspace of \tilde{H}_w . For the converse, if \tilde{H}_v is isometrically equivalent to a Hilbert subspace of \tilde{H}_w , then v can be represented in $L^2(\mathbb{R}, \mu_w)$ by some function, and therefore, $\mu_u \ll \mu_v$. \square

Remark 6.2. Recall that if $G \subseteq \tilde{H}$ is small, $S(G)$ denotes the set of (1) Galois types over G .

Theorem 6.3. Let $p, q \in S(\emptyset)$ and let $v, w \in \tilde{\mathcal{H}}$ such that $v \models p$ and $w \models q$, and $\mu_v \ll \mu_w$. Then, $d(p, q) = \|\mu_w - \mu_v\|$

Proof. If $\mu_u \ll \mu_v$, by Theorem 6.1, there exist $v' \models tp(v/\emptyset)$ such that $\tilde{H}_{v'} \leq \tilde{H}_w$ and there exists $f \in L^1(\sigma(N), \mu_w)$ such that $d\mu_v = f d\mu_w$. Then $d|\mu_w - \mu_v| = |1 - f| d\mu_w$ and therefore $d(p, q) = \|\mu_w - \mu_v\|$. \square

Theorem 6.4. Let $p, q \in S(\emptyset)$ and let $v, w \in \tilde{H}$ be such that $v \models p$ and $w \models q$, and $\mu_v \perp \mu_w$. Then, $d(p, q) = \sqrt{\|\mu_v\|^2 + \|\mu_w\|^2}$

Proof. If $\mu_v \perp \mu_w$, by Theorem 6.1, neither \tilde{H}_v is not isometrically isomorphic to a Hilbert subspace of \tilde{H}_w nor \tilde{H}_w is isometrically isomorphic to a Hilbert subspace of \tilde{H}_v . Then we can assume $\tilde{H}_v \perp \tilde{H}_w$ and therefore, $d(p, q) = \|v - w\| = \sqrt{\|v\|^2 + \|w\|^2} = \sqrt{\|\mu_v\|^2 + \|\mu_w\|^2}$. \square

Theorem 6.5. *Let $p, q \in S(\emptyset)$ and let $v, w \in \tilde{\mathcal{H}}$ be such that $v \models p$ and $w \models q$, and $\mu_w = \mu_w^\parallel + \mu_w^\perp$ according to Lebesgue decomposition theorem. Then, $d(p, q) = \sqrt{\|\mu_v - \mu_w^\parallel\|^2 + \|\mu_w^\perp\|^2}$*

Proof. By Theorem 6.3 and Theorem 6.4. \square

Theorem 6.6. *Let $G \subseteq \tilde{H}$ be small, let $p, q \in S(G)$ and let $v, w \in \tilde{H}$ be such that $u \models p$ and $v \models q$. Then,*

$$d(p, q) = \sqrt{[P(v) - P_G(w)]^2 + d^2(\text{gatp}(P_G^\perp v / \emptyset), \text{gatp}(P_G^\perp w / \emptyset))}$$

Proof. By Theorems 3.16 \square

Corollary 6.7. *Let $G \subseteq \tilde{H}$ then $\text{dens}[S_1(F)] \leq |F| \times 2^{\aleph_0}$*

Proof. Clear from Theorem 3.16, Theorem 3.16 and Theorem 6.6. \square

Theorem 6.8. $\mathcal{K}_{(H, \Gamma_Q)}$ is κ -stable for $\kappa \geq |\sigma|$.

Proof. Clear from Corollary 6.7. \square

Definition 6.9. A MAEC \mathcal{K} is said to be \aleph_0 -stable up to perturbations if for every pair of separable structure $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$, every type $p \in S(\mathcal{M})$ and every $\epsilon > 0$, there is a separable structure \mathcal{N}' and an ϵ -perturbation $f : \mathcal{N} \rightarrow \mathcal{N}'$ such that p is realized in \mathcal{N}' and f is a (0)isomorphism over \mathcal{M} .

Theorem 6.10. $\mathcal{K}_{(H, \Gamma_Q)}$ is \aleph_0 -stable up to perturbations.

Proof. Let $(H_0, Q_0) \prec (H_1, \Gamma_{Q_1}) \in \mathcal{K}_{(H, \Gamma_Q)}$, and let $p \in S(H_0)$. Let $v \in \tilde{H}$ be a realization of p in the monster model. Since $(H_0, Q_0) \oplus (L^2(\mathbb{R}, \mu_{v_e}), M_{f_{v_e}})$ and (H_1, Γ_{Q_1}) are separable and spectrally equivalent, by Theorem 2.29, they are approximately uniformly equivalent and therefore there is an ϵ -perturbation relating (H_1, Γ_{Q_1}) and $(H_0, Q_0) \oplus (L^2(\mathbb{R}, \mu_{v_e}), M_{f_{v_e}})$. \square

7. SPECTRAL INDEPENDENCE

In this section we define an independence relation in $\mathcal{K}_{(H, \Gamma_Q)}$, called *spectral independence*. Theorem 7.6 states that this relation has the same properties as non-forking for superstable firstorder theories, while Theorem 7.8 and Theorem 7.9 state that this relation characterize non-splitting.

Definition 7.1. Let $v \in \tilde{H}$ and let $F, G \subseteq \tilde{H}$. We say that v is *spectrally independent* from G over F if $P_{\text{acl}(F)}v = P_{\text{acl}(F \cup G)}v$ and denote it $v \perp_F^* G$.

Remark 7.2. Let $v, w \in \tilde{H}$. Then v is independent from w over \emptyset if and only if $\tilde{H}_{v_e} \perp \tilde{H}_{w_e}$ and denote it $v \perp_\emptyset^* w$.

Remark 7.3. Let $v, w \in \tilde{H}$. Let $G \subseteq \tilde{H}$ be small. Then v is independent from w over G if and only if $\tilde{H}_{P_{\text{acl}(G)}^\perp(v)} \perp \tilde{H}_{P_{\text{acl}(G)}^\perp(w)}$ and denote it $v \perp_G^* w$.

Remark 7.4. Let $\bar{v} \in H^n$ and $E, F \subseteq H$. Then $\bar{v} \downarrow_E^* F$ if and only if for every $j = 1, \dots, n$ $v_j \downarrow_E^* F$ that is, for all $j = 1, \dots, n$ $P_{acl(E)}(v_j) = P_{acl(E \cup F)}(v_j)$

Theorem 7.5. Let $F \subseteq G \subseteq H$, $p \in S_n(F)$ $q \in S_n(G)$ and $\bar{v} = (v_1, \dots, v_n)$, $\bar{w} = (w_1, \dots, w_n) \in H^n$ be such that $p = tp(\bar{v}/F)$ and $q = tp(\bar{w}/G)$. Then q is an extension of p such that $\bar{w} \downarrow_F^* G$ if and only if the following conditions hold:

- (1) For every $j = 1, \dots, n$, $P_{acl(F)}(v_j) = P_{acl(G)}(w_j)$
- (2) For every $j = 1, \dots, n$, $\mu_{P_{acl(F)}^\perp} v_j = \mu_{P_{acl(G)}^\perp} w_j$

Proof. Clear from Theorem 3.16 and Remark 7.3 □

Theorem 7.6. \downarrow^* satisfies:

- (1) Local character.
- (2) Finite character.
- (3) Transitivity of independence
- (4) Symmetry
- (5) Existence
- (6) Stationarity

Proof. By Remark 7.4, to prove local character, finite character and transitivity it is enough to show them for the case of a 1-tuple.

Local character: Let $v \in H$ and $G \subseteq \tilde{H}$. Let $w = (P_{acl(G)}(v))_e$. Then there exist a sequence of $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, a sequence $(f_1^k, \dots, f_{l_k}^k)_{k \in \mathbb{N}}$ of finite tuples of bounded Borel functions of \mathbb{R} and a sequence of finite tuples $(e_1^k, \dots, e_{l_k}^k)_{k \in \mathbb{N}} \subseteq G$ such that if $w_k := \sum_{j=1}^{l_k} f_j^k(\tilde{Q}) e_j^k$ for $k \in \mathbb{N}$, then $w_k \rightarrow w$ when $k \rightarrow \infty$. Let $E_0 = \{e_j^k \mid j = 1, \dots, l_k \text{ and } k \in \mathbb{N}\}$. Then $v \downarrow_{E_0}^* E$ and $|E_0| = \aleph_0$.

Finite character: We show that for $v \in H$, $E, F \subseteq \tilde{H}$, $v \downarrow_E^* F$ if and only if $v \downarrow_E^* F_0$ for every finite $F_0 \subseteq F$. The left to right direction is clear. For right to left, suppose that $v \not\downarrow_E^* F$. Let $w = P_{acl(E \cup F)}(v) - P_{acl(E)}(v)$. Then $w \in \text{acl}(E \cup F) \setminus \text{acl}(E)$.

As in the proof of local character, there exist a sequence of pairs $(l_k, n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}^2$, a sequence $(g_1^k, \dots, g_{l_k+n_k}^k)_{k \in \mathbb{N}}$ of finite tuples of bounded Borel functions on \mathbb{R} , and a sequence of finite tuples $(e_1^k, \dots, e_{l_k}^k, f_1^k, \dots, f_{n_k}^k)_{k \in \mathbb{N}}$ such that $(e_1^k, \dots, e_{l_k}^k) \subseteq E$, $(f_1^k, \dots, f_{n_k}^k)_{k \in \mathbb{N}} \subseteq F$ and if $w_k := \sum_{j=1}^{l_k} g_j^k(\tilde{Q}) e_j^k + \sum_{j=1}^{n_k} g_{l_k+j}^k(\tilde{Q}) f_j^k$ for $k \in \mathbb{N}$, then $w_k \rightarrow w$ when $k \rightarrow \infty$.

If $v \not\downarrow_E^* F$, then $w = P_{acl(E \cup F)}(v) - P_{acl(E)}(v) \neq 0$. For $\epsilon = \|w\| > 0$ there is k_ϵ such that if $k \geq k_\epsilon$ then $\|w - w_k\| < \epsilon$. Let $F_0 := \{f_1^1, \dots, f_{n_{k_\epsilon}}^{n_{k_\epsilon}}\}$. Then F_0 is a finite subset such that $v \not\downarrow_{E \cup F_0}^* F_0$.

Transitivity of independence: Let $v \in H$ and $E \subseteq F \subseteq G \subseteq H$. If $v \downarrow_E^* G$ then $P_{acl(E)}(v) = P_{acl(G)}(v)$. It is clear that $P_{acl(E)}(v) = P_{acl(F)}(v) = P_{acl(G)}(v)$ so $v \downarrow_E^* F$ and $v \downarrow_F^* G$. Conversely, if $v \downarrow_E^* F$ and $v \downarrow_F^* G$, we have that $P_{acl(E)}(v) = P_{acl(F)}(v)$ and $P_{acl(F)}(v) = P_{acl(G)}(v)$. Then $P_{acl(E)}(v) = P_{acl(G)}(v)$ and $v \downarrow_E^* G$.

Symmetry: It is clear from Remark 7.3.

Invariance: Let U be an automorphism of $(\tilde{H}, \Gamma_{\tilde{Q}})$. Let $\bar{v} = (v_1, \dots, v_n), \bar{w} = (w_1, \dots, w_n) \in \tilde{H}^n$ and $G \subseteq \tilde{H}$ be such that $\bar{v} \downarrow_G^* \bar{w}$. By Remark 7.3, this

means that for every $j, k = 1, \dots, n$ $\tilde{H}_{P_{acl(G)}^\perp(v_j)} \perp \tilde{H}_{P_{acl(G)}^\perp(w_k)}$. It follows that for every $j, k = 1, \dots, n$ $\tilde{H}_{P_{acl(UG)}^\perp(Uv_j)} \perp \tilde{H}_{P_{acl(UG)}^\perp(Uw_k)}$ and, again by Remark 7.3, $Uv \downarrow_{acl(UG)}^* Uw$.

Existence: Let $F \subseteq G \subseteq \tilde{H}$ be small sets. We show, by induction on n , that for every $p \in S_n(F)$, there exists $q \in S_n(G)$ such that q is an \downarrow^* -independent extension of p .

Case $n = 1$: Let $v \in \tilde{H}$ be such that $p = tp(v/F)$ and let $(H', Q') \in \mathcal{K}_{(H, \Gamma_Q)}$ be a structure containing v and G . Define

$$H'' := H' \oplus L^2(\mathbb{R}, \mu_{(P_{acl(F)}^\perp v)_e}),$$

$$Q'' := Q' \oplus M_{f_{(P_{acl(F)}^\perp v)_e}}$$

and

$$v' := P_{acl(F)}v + P_{acl(F)}^\perp v_d + (1)_{\sim \mu_{(P_{acl(F)}^\perp v)_e}}$$

Then $(H'', Q'') \in \mathcal{K}_{(H, \Gamma_Q)}$, $v' \in H''$ and, by Theorem 7.5, the type $gatp(v'/G)$ is a \downarrow^* -independent extension of $tp(v/F)$.

Induction step: Now, let $\bar{v} = (v_1, \dots, v_n, v_{n+1}) \in \tilde{H}^{n+1}$. By induction hypothesis, there are $v'_1, \dots, v'_n \in H$ such that $gatp(v'_1, \dots, v'_n/G)$ is a \downarrow^* -independent extension of $gatp(v_1, \dots, v_n/F)$. Let U be a monster model automorphism fixing F pointwise such that for every $j = 1, \dots, n$, $U(v_j) = v'_j$. Let $v'_{n+1} \in \tilde{H}$ be such that $gatp(v'_{n+1}/Gv'_1 \cdots v'_n)$ is a \downarrow^* -independent extension of $gatp(U(v_{n+1})/Fv'_1, \dots, v'_n)$. Then, by transitivity, $gatp(v'_1, \dots, v'_n, v'_{n+1}/G)$ is a \downarrow^* -independent extension of $gatp(v_1, \dots, v_n, v_{n+1}/F)$.

Stationarity: Let $F \subseteq G \subseteq \tilde{H}$ be small sets. We show, by induction on n , that for every $p \in S_n(F)$, if $q \in S_n(G)$ is a \downarrow^* -independent extension of p to G then $q = p'$, where p' is the \downarrow^* -independent extension of p to G built in the proof of existence.

Case $n = 1$: Let $v \in H$ be such that $p = gatp(v/F)$, and let $q \in S(G)$ and $w \in H$ be such that $w \models q$. Let v' be as in previous item. Then, by Theorem 7.5 we have that:

$$(1) \ P_{acl(F)}v = P_{acl(G)}v' = P_{acl(G)}w =$$

$$(2) \ \mu_{P_{acl(F)}^\perp v} = \mu_{P_{acl(G)}^\perp w} = \mu_{P_{acl(G)}^\perp v'}$$

This means that $P_{acl(G)}v' = P_{acl(G)}w$, $\mu_{P_{acl(G)}^\perp v'} = \mu_{P_{acl(G)}^\perp w}$ and, therefore $q = tp(v'/G) = p'$.

Induction step: Let $\bar{v} = (v_1, \dots, v_n, v_{n+1})$, $\bar{v}' = (v'_1, \dots, v'_n, v'_{n+1})$ and $\bar{w} = (w_1, \dots, w_n) \in \tilde{H}$ be such that $\bar{v} \models p$, $\bar{v}' \models p'$ and $\bar{w} \models q$. By transitivity, we have that $gatp(v'_1, \dots, v'_n/G)$ and $gatp(w_1, \dots, w_n/G)$ are \downarrow^* -independent extensions of $gatp(v_1, \dots, v_n/F)$. By induction hypothesis, $gatp(v'_1, \dots, v'_n/G) = gatp(w_1, \dots, w_n/G)$. Let U be a monster model automorphism fixing F pointwise such that for every $j = 1, \dots, n$, $U(v_j) = v'_j$ and let U' a monster model automorphism fixing G pointwise such that for every $j = 1, \dots, n$, $U'(v'_j) = w'_j$. Again by transitivity,

$$gatp(U^{-1}(v'_{n+1})/Gv_1 \cdots v_n)$$

and

$$\text{gatp}((U' \circ U)^{-1}(w_{n+1})/Gv_1, \dots, v_n)$$

are \downarrow^* -independent extensions of $\text{gatp}(v_{n+1}/Fv_1, \dots, v_n)$.

By the case $n = 1$,

$$\text{gatp}(U^{-1}(v'_{n+1})/Gv_1 \cdots v_n) = \text{gatp}((U' \circ U)^{-1}(w_{n+1})/Gv_1, \dots, v_n)$$

and therefore

$$p' = \text{gatp}(v'_1, \dots, v'_n v'_{n+1}/G) = \text{gatp}(w_1, \dots, w_n, w_{n+1}/G) = q.$$

□

Definition 7.7. Let \mathcal{K} be an homogeneous MAEC with monster model \mathcal{M} . Let $B \subseteq A \subseteq M$ and let $a \in M$. The type $\text{gatp}(a/A)$ is said to *split* over B if there are $b, c \in A$ such that

$$\text{gatp}(b/B) = \text{gatp}(c/B)$$

but

$$\text{gatp}(b/Ba) \neq \text{gatp}(c/Ba)$$

Theorem 7.8. Let $v \in \tilde{H}$ and let $F \subseteq G \subseteq \tilde{H}$. If $\text{gatp}(v/G)$ splits over F then $v \not\downarrow_F^* G$.

Proof. If $\text{gatp}(v/G)$ splits over F , then there are two vectors w_1 and $w_2 \in G$ such that $\text{gatp}(w_1/F) = \text{gatp}(w_2/F)$ but $\text{gatp}(w_1/Fv) \neq \text{gatp}(w_2/Fv)$. Then, either $\text{gatp}(P_{\text{acl}(Fv)}^\perp w_1/\emptyset) \neq \text{gatp}(P_{\text{acl}(Fv)}^\perp w_2/\emptyset)$ or $P_{\text{acl}(Fv)} w_1 \neq P_{\text{acl}(Fv)} w_2$. Let us consider each case:

Case $\text{gatp}(P_{\text{acl}(Fv)}^\perp w_1/\emptyset) \neq \text{gatp}(P_{\text{acl}(Fv)}^\perp w_2/\emptyset)$: Since

$$P_{\text{acl}(Fv)}^\perp w_1 = P_{\text{acl}(F)}^\perp w_1 - P_{P_{\text{acl}(F)}^\perp v_e} w_1$$

and

$$P_{\text{acl}(Fv)}^\perp w_2 = P_{\text{acl}(F)}^\perp w_2 - P_{P_{\text{acl}(F)}^\perp v_e} w_2,$$

this means that

$$\text{gatp}(P_{P_{\text{acl}(F)}^\perp v_e} w_1/\emptyset) \neq \text{gatp}(P_{P_{\text{acl}(F)}^\perp v_e} w_2/\emptyset)$$

So, either $P_{P_{\text{acl}(F)}^\perp v_e} w_1 \neq 0$ or $P_{P_{\text{acl}(F)}^\perp v_e} w_2 \neq 0$. Let us suppose without loss of generality that $P_{P_{\text{acl}(F)}^\perp v_e} w_1 \neq 0$. Then $P_{w_1}(P_{\text{acl}(F)}^\perp v_e) \neq 0$, which implies that $P_{\text{acl}(F)} v \neq P_{\text{acl}(Fw_1)} v$. That is, $v \not\downarrow_F^* w_1$ and by transitivity, $v \not\downarrow_F^* G$.

Case $P_{\text{acl}(Fv)} w_1 \neq P_{\text{acl}(Fv)} w_2$: Since

$$P_{\text{acl}(Fv)} w_1 = P_{\text{acl}(F)} w_1 + P_{P_{\text{acl}(F)}^\perp v_e} w_1$$

and

$$P_{\text{acl}(Fv)} w_2 = P_{\text{acl}(F)} w_2 + P_{P_{\text{acl}(F)}^\perp v_e} w_2,$$

this means that $P_{P_{\text{acl}(F)}^\perp v_e} w_1 \neq P_{P_{\text{acl}(F)}^\perp v_e} w_2$ and, therefore either $P_{P_{\text{acl}(F)}^\perp v_e} w_1 \neq 0$ or $P_{P_{\text{acl}(F)}^\perp v_e} w_2 \neq 0$. As in previous item, this implies that $v \not\downarrow_F^* G$.

□

Theorem 7.9. Let $v \in \tilde{H}$ and $F \subseteq G \subseteq \tilde{H}$ such that $F = \text{acl}(F)$ and B is $|A|$ -saturated. If $v \not\downarrow_F^* G$, then v splits over F .

Proof. If $v \not\downarrow_F^* G$ then $w := P_G v - P_F v \neq 0$ and $w \perp F$. Since G is $|F|$ -saturated, there is $w' \in G$ such that $\text{gatp}(w/F) = \text{gatp}(w'/F)$ and $w' \perp P_G v$. Since $\langle v | w \rangle \neq 0$, $P_v w \neq 0$, while $P_v w' = 0$. \square

Definition 7.10. Let $\epsilon > 0$, $v \in \tilde{H}$ and let $F, G \subseteq \tilde{H}$. We say that v is ϵ -spectrally independent from G over F if $P_{\text{acl}(F \cup G)} v - P_{\text{acl}(F)} v \leq \epsilon$ and denote it $v \downarrow_F^\epsilon G$.

Theorem 7.11. The relation \downarrow^ϵ satisfies the following properties:

Local character: Let $v \in H$, $G \subseteq \tilde{H}$ and $\epsilon > 0$. Then there is a finite $G_0 \subseteq G$ such that $v \downarrow_{G_0}^\epsilon G$.

Transitivity of independence: Let $v \in H$ and $D \subseteq E \subseteq F \subseteq G \subseteq H$. If $v \downarrow_D^\epsilon G$ then $v \downarrow_E^\epsilon F$.

Proof. **Local character:** Let $v \in H$, $G \subseteq \tilde{H}$ and $\epsilon > 0$. Let $w, (l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, $(e_1^k, \dots, e_{l_k}^k)_{k \in \mathbb{N}} \subseteq G$, $(f_1^k, \dots, f_{l_k}^k)_{k \in \mathbb{N}}$ and w_k for $k \in \mathbb{N}$ be as in the proof of local character of \downarrow^* in Theorem 7.6. Since $w_k \rightarrow w$ when $k \rightarrow \infty$, there is a $k_1 \in \mathbb{Z}$ such that $\|w_k - w\| < \epsilon$ for all $k \geq k_1$. Let $G_o := \{e_j^k \mid j = 1, \dots, l_k \text{ and } k \leq k_1\}$. Then, $v \downarrow_{G_o}^* G$.

Transitivity of independence: Let $v \in H$ and $D \subseteq E \subseteq F \subseteq G \subseteq H$ and $\epsilon > 0$. If $v \downarrow_D^\epsilon G$ then $\epsilon \geq P_{\text{acl}(D \cup G)} v - P_{\text{acl}(D)} v = P_{\text{acl}(G)} v - P_{\text{acl}(D)} v \geq P_{\text{acl}(F)} v - P_{\text{acl}(E)} v$. Therefore $v \downarrow_E^\epsilon F$. \square

Definition 7.12. Let $\bar{v} = (v_1, \dots, v_n) \in H^n$ and $G \subseteq H$. A *canonical base* for the type $\text{gatp}(\bar{v}/G)$ is a set $F \subseteq H_G$ such that $\bar{v} \downarrow_F^* G$ and $|F|$ is minimal.

Theorem 7.13. Let $\bar{v} = (v_1, \dots, v_n) \in H^n$ and $G \subseteq H$. Then $\text{Cb}(\text{gatp}(\bar{v}/G)) := \{(P_G v_1, \dots, P_G v_n)\}$ is a canonical base for the type $\text{gatp}(\bar{v}/G)$.

Proof. First of all, we consider the case of a 1-tuple. By Theorem 7.5 $\text{gatp}(v/G)$ does not fork over $\text{Cb}(\text{gatp}(v/G))$. Let $(v_k)_{k < \omega}$ a Morley sequence for $\text{gatp}(v/G)$. We have to show that $P_G v \in \text{dcl}((v_k)_{k < \omega})$. By Theorem 7.5, for every $k < \omega$ there is a vector w_k such that $v_k = P_G v + w_k$ and $w_k \perp \text{acl}(\{P_G v\} \cup \{w_j \mid j < k\})$. This means that for every $k < \omega$, $w_k \in H_e$ and for all $j, k < \omega$, $H_{w_j} \perp H_{w_k}$. For $k < \omega$, let $v'_k := \frac{v_1 + \dots + v_k}{n} = P_G v + \frac{w_1 + \dots + w_k}{n}$. Then for every $k < \omega$, $v'_k \in \text{dcl}((v_k)_{k < \omega})$. Since $v'_k \rightarrow P_e v$ when $k \rightarrow \infty$, we have that $P_G v \in \text{dcl}((v_k)_{k < \omega})$.

For the case of a general n -tuple, by Remark 7.4, it is enough to repeat previous argument in every component of \bar{v} . \square

8. ORTHOGONALITY AND DOMINATION

In this section, we characterize domination, orthogonality of types in terms of absolute continuity and mutual singularity between spectral measures. This is done in Corollary 8.2 and Corollary 8.5.

Theorem 8.1. Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\mu_{v_e} \perp \mu_{w_e}$.

Proof. $p \perp^a q$ if and only if $\tilde{H}_{v'_e} \perp \tilde{H}_{w'_e}$ for all $v'_e \models p$ and $w'_e \models q$. By Lebesgue decomposition theorem $\mu_{w_e} = \mu_{v_e}^\parallel + \mu_{v_e}^\perp$ where, $\mu_{v_e}^\parallel < \mu_{v_e}$ and $\mu_{v_e}^\perp \perp \mu_{v_e}$. $\mu_{v_e}^\parallel \neq 0$ if and only if there is a choice of $v' \models p$ and $w' \models q$ such that $\tilde{H}_{v'_e} \cap \tilde{H}_{w'_e} \neq \{0\}$ and therefore $\tilde{H}_{v'_e} \not\perp \tilde{H}_{w'_e}$. \square

Corollary 8.2. *Let $G \subseteq \tilde{H}$ be small. Let $p, q \in S_1(G)$, let $v \models p$ and $w \models q$. Then, $p \perp_G^a q$ if and only if $\mu_{P_G^\perp v_e} \perp \mu_{P_G^\perp w_e}$*

Proof. Clear from Theorem 8.1. \square

Corollary 8.3. *Let $G \subseteq H$ be small. Let $p, q \in S_1(G)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Proof. Clear from Corollary 8.2. \square

Theorem 8.4. *Let $p, q \in S_1(\emptyset)$, let $v \models p$ and $w \models q$. Then, $p \triangleright_\emptyset q$ if and only if $\mu_{v_e} \gg \mu_{w_e}$.*

Proof. Suppose $p \triangleright_\emptyset q$. Suppose that v and w are such that if $v \downarrow_\emptyset^* G$ then $w \downarrow_\emptyset^* G$ for every $G \subseteq \tilde{H}$. Then for every G if $\tilde{H}_{v_e} \perp \tilde{H}_G$ then $\tilde{H}_{w_e} \perp \tilde{H}_G$. This means $\tilde{H}_{w_e} \subseteq \tilde{H}_{v_e}$ and \tilde{H}_{w_e} is unitarily equivalent to some Hilbert subspace of \tilde{H}_{v_e} and by Theorem 6.1 $\mu_{w_e} < \mu_{v_e}$. \square

Corollary 8.5. *Let E, F , and G be small subsets of \tilde{H} and $p \in S_1(F)$ and $q \in S_1(G)$ two stationary types. Then $p \triangleright_E q$ if and only if there exist $v, w \in \tilde{\mathcal{H}}$ such that $\text{gatp}(v/E)$ is a non-forking extension of p , $\text{gatp}(w/E)$ is a non-forking extension of q and $\mu_{P_{\text{acl}(F)}^\perp v} \gg \mu_{P_{\text{acl}(F)}^\perp w}$.*

Proof. Clear from previous theorem. \square

REFERENCES

- [1] N.I. Akhiezer, I.M. Glazman, *Theory of linear operators in Hilbert Space vols.I and II*. Pitman Advanced Publishing Program, 1981.
- [2] W. Arveson, *A short course on spectral theory* Springer Verlag.
- [3] C. Argoty, *Model theory of a non-degenerate representation of a unital C^* -algebra*. Submitted, available in <http://arxiv.org/abs/1010.6188>
- [4] C. Argoty, I. Ben Yaacov, *Model theory Hilbert spaces expanded with a bounded normal operator*. Available in <http://www.maths.manchester.ac.uk/logic/mathlogaps/preprints/hilbertnormal1.pdf>
- [5] C. Argoty, A. Berenstein, *Hilbert spaces expanded with an unitary operator*. Math. Log. Quart. 55, No. 1, 37 - 50 (2009)
- [6] I. Ben Yaacov, *On perturbations of continuous structures*. Jour. Math. Log. 8, No. 2, 225-249 (2008).
- [7] Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoe Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.). London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315-427.
- [8] Itai Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*. Jour. Math. Log. 8, No. 2, 197-223(2008).
- [9] Itai Ben Yaacov, Alexander Usvyatsov and Moshe Zadka, *Generic automorphism of a Hilbert space*, preprint.
- [10] D. Berg, *An extension the Weyl-Von Neumann theorem to normal operators*. Trans. Amer. Math. Soc. Volume 160, 1971.
- [11] Alexander Berenstein, and Steven Buechler, *Simple stable homogeneous expansions of Hilbert spaces*. Annals of of pure and Applied Logic. Vol. 128 (2004) pag 75-101.
- [12] Steven Buechler, *Essential stability theory*. Springer Verlag, 1991.
- [13] K. Davidson, *C^* -algebras by example*. Fields institute monographs, American Mathematical Society, 1996.
- [14] R. Dautray, L. Lions, *Mathematical analysis and numerical methods for science and technology*, volume 3.
- [15] N.Dunford,J. Schwarz, *Linear operators*. John Wiley & Sons, 1971.

- [16] C. Ward Henson and Hernando Tellez, *Algebraic closure in continuous logic*. Revista Colombiana de Matemáticas Vol 41 [Especial] (2007). 279–285.
- [17] Å. Hirvonen, *Categoricity in homogeneous complete metric spaces*. Ph.D thesis, University of Helsinki. Available at <http://www.helsinki.fi/asaekman/paperit/categoricityv2.pdf>
- [18] Å. Hirvonen, *Metric Abstract Elementary Classes with Perturbations*.
- [19] José Iovino, *Stable theories in functional analysis* University of Illinois Ph.D. Thesis, 1994.
- [20] L. Liusternik and V. Sobolev, *Elements of Functional Analysis*. Frederic Ungar Publishing Co., New York, 1961
- [21] M. Reed, B. Simon, *Methods of modern mathematical physics* volume I: *Functional analysis, revised and enlarged edition*. Academic Press, 1980.
- [22] W. Rudin *real and complex analysis* McGraw Hill
- [23] S. Shelah, *Classification of nonelementary classes, II, Abstract Elementary Classes*. In Classification Theory (Chicago IL 1985), volume 1292 of Lecture Notes in Mathematics, pages 419-497. Springer, Berlin, 1987. Proceedings of the USA-Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.
- [24] A. Villaveces, P. Zambrano, *Notions of independence for metric abstract elementary classes*. Report No 27, 2009/2010 of Mittag-Leffler Institute (Sweden).
- [25] A. Villaveces, P. Zambrano, *Around Superstability in Metric Abstract Elementary Classes: Limit Models and r -Towers*. Report No 33, 2009/2010 of Mittag-Leffler Institute (Sweden).

CAMILO ARGOTY, UNIVERSIDAD SERGIO ARBOLEDA, DEPARTAMENTO DE MATEMÁTICAS, CALLE 74 # 14-14, BOGOTÁ, COLOMBIA